Simple mathematics of QPOs

Marek Abramowicz
Omer Blaes, Jiří Horák, Włodek Kluźniak,
Paola Rebusco, and Eva Šrámková

Lagrangian method

Eulerian and Lagrangian perturbations,

$$\delta Q(\boldsymbol{x},t) \;\equiv\; Q(\boldsymbol{x},t) - Q_0(\boldsymbol{x},t),$$
$$\Delta Q(\boldsymbol{x},t) \;\equiv\; Q(\boldsymbol{x}+\boldsymbol{\xi},t) - Q_0(\boldsymbol{x},t)$$

Lagrangian perturbations:

$$\Delta \boldsymbol{v} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t}, \qquad \frac{\Delta\rho}{\rho} = \frac{1-\mathcal{J}}{\mathcal{J}}, \qquad \mathcal{J} = \det\left(\delta^i_j + \frac{\partial\xi^i}{\partial x_j}\right)$$

Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\rho\left|\boldsymbol{v} + \frac{\partial\boldsymbol{\xi}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{\xi}\right|^2 - p\frac{\mathcal{J}^{1-\gamma}}{\gamma-1} - \rho\Phi(\boldsymbol{x}+\boldsymbol{\xi}).$$

Perturbative expansion

Expansion of the Lagrangian density, $\mathcal{L}^{(n)} \sim \xi^n$

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)} + \mathcal{L}^{(5)} + \mathcal{O}(\xi^6),$$

$$\frac{\mathrm{D}^2\xi_i}{\mathrm{D}t^2} - \frac{1}{\rho}(\gamma - 1)\nabla_i(p\nabla_k\xi^k)$$
$$-\frac{1}{\rho}\nabla_k(p\nabla_i\xi^k) + \xi^k\nabla_k\nabla_i\Phi = a_i(\xi)$$

- LHS $\rightarrow$ Linear terms
- RHS $\rightarrow$ Nonlinear accelerations

Linear Modes

- Solution of the linear problem:

$$\boldsymbol{\xi}(\boldsymbol{x}, t) \equiv \boldsymbol{\xi}(\boldsymbol{x}) \exp\left[\mathrm{i}\omega t\right]$$

- Eigenfunctions $\boldsymbol{\xi}_\alpha(\boldsymbol{x})$ form (non)orthogonal basis with respect to the scalar product,

$$\left\langle \xi, \xi' \right\rangle \equiv \int_V \rho \boldsymbol{\xi} \cdot \boldsymbol{\xi}' \mathrm{d}V$$

- General perturbation (solution of the nonlinear equation) can be expressed

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \sum_\alpha c_\alpha(t) \boldsymbol{\xi}_\alpha(\boldsymbol{x}) + \bar{c}_\alpha(t) \bar{\boldsymbol{\xi}}_\alpha(\boldsymbol{x})$$

Non-linear oscillators

Governing equation for $c_\alpha(t)$

$$\frac{\mathrm{d}c_\alpha}{\mathrm{d}t} + \mathrm{i}\omega_\alpha c_\alpha = \frac{\mathrm{i}}{b_\alpha}\langle \boldsymbol{\xi}_\alpha, \boldsymbol{a}\rangle$$

or with $c_\alpha = -i\,\omega_\alpha^0\,X_\alpha + \delta\dot{X}_\alpha$ one gets for $\alpha = U, D$,

$$\begin{aligned}
\ddot{X}_U + (\omega_U^0)^2 X_U &= \mathcal{F}_U(X_U, \dot{X}_U, X_L, \dot{X}_L), \\
\ddot{X}_L + (\omega_L^0)^2 X_L &= \mathcal{F}_L(X_U, \dot{X}_U, X_L, \dot{X}_L).
\end{aligned}$$

i.e. equations for coupled nonlinear oscillators.
Rebusco and Horák (multiple scales method).

Main results

[1] Frequencies depend on amplitudes:

$$\begin{aligned} \nu_{\mathrm{U}} &= \nu^0_{\mathrm{U}} + \nu^0_{\mathrm{U}}(C_{UU}\alpha^2_U + C_{UD}\alpha^2_D) + \mathcal{O}^3(\alpha), \\ \nu_{\mathrm{D}} &= \nu^0_{\mathrm{D}} + \nu^0_{\mathrm{D}}(C_{DU}\alpha^2_U + C_{DD}\alpha^2_D) + \mathcal{O}^3(\alpha). \end{aligned}$$

[2] Energy of oscillation is constant:

$$\begin{aligned} \mathcal{E}_0 &= \alpha^2_U + \mathcal{K}\alpha^2_D = \mathrm{constant} + \mathcal{O}^3(\alpha), \\ \alpha^2_U &= s\,\mathcal{E}_0, \quad \alpha^2_D = (1-s)\frac{\mathcal{E}_0}{\mathcal{K}} \end{aligned}$$

[2] $\rightarrow$ [3] Amplitudes are **(anti)correlated.**

Correlation of frequencies

The amplitudes are correlated also when,

$$\mathcal{E} = \mathcal{E}_0 + \Delta\mathcal{E}(s),$$

or when the energy is a slowly varying function of s, or when there is *any* physical connection between excitation and damping. From the correlation of amplitudes [2], and from the frequency-amplitude depenence [1], the correlation of frequencies follows,

$$\nu_{\rm U} = \nu_{\rm U}^0 + F(s), \quad \nu_{\rm D} = \nu_{\rm D}^0 + G(s).$$

Functions $F(s)$ and $G(s)$ are *known* for a given system.

The Bursa line

For a weak non-linearity and a weak coupling,

$$F = F'\,s + \mathcal{O}(s^2), \quad G = G'\,s + \mathcal{O}(s^2).$$

We define,

$$X \equiv G'/F', \quad A = X, \quad B = \nu^0_{\rm U} - X\,\nu^0_{\rm D},$$

and get from this simple mathematics the Bursa line:

$$\nu_{\rm U} = A\,\nu_{\rm D} + B,$$

i.e. a linear frequency-frequency correlation.

The slopes of Bursa lines

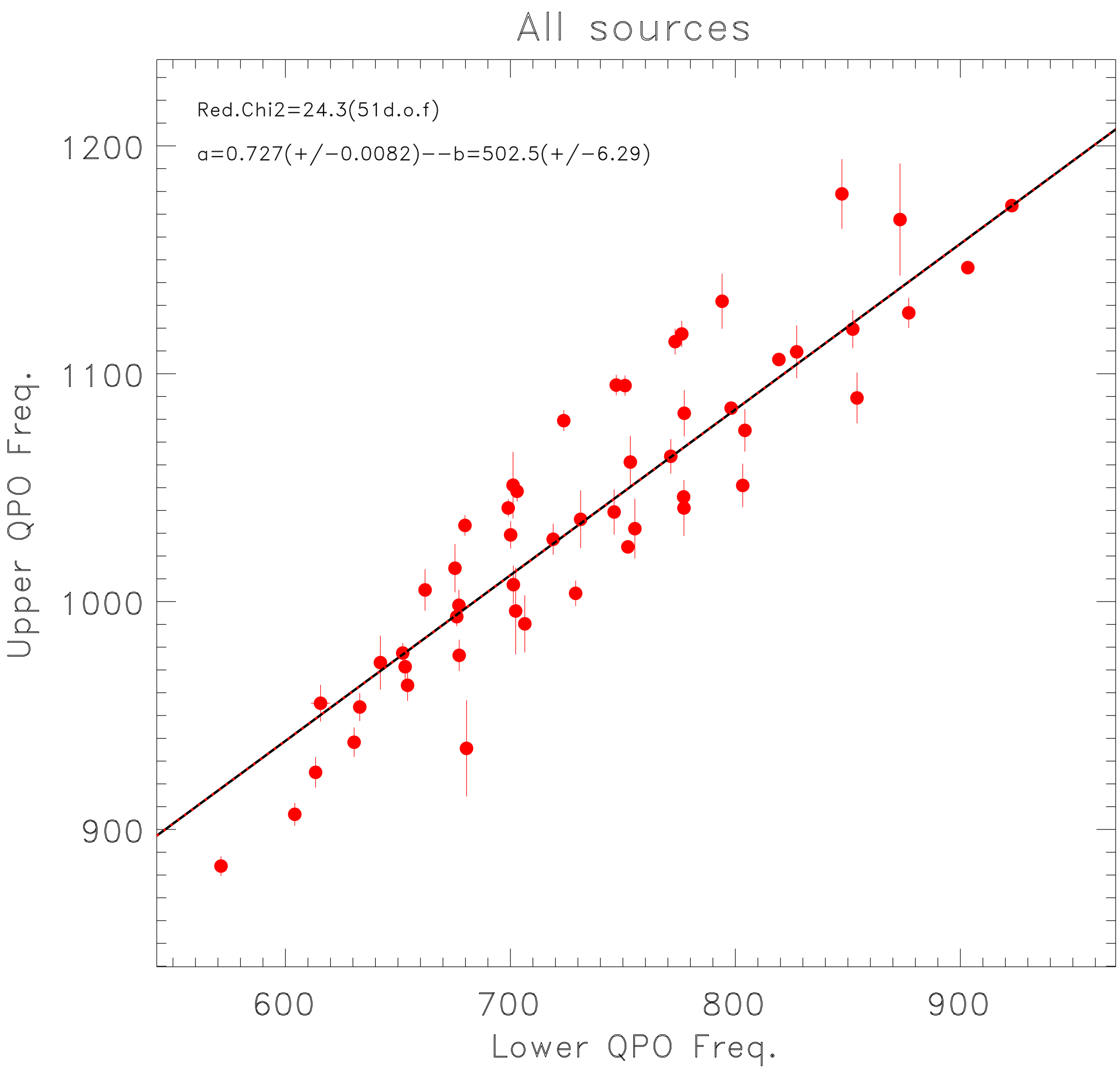
Take a conservative system as an example.

$$A = \frac{\nu^0_{\mathrm{U}}}{\nu^0_{\mathrm{D}}}\Gamma, \quad \Gamma = \frac{(C_{UU} - C_{UD}/\mathcal{K})}{(C_{DU} - C_{DD}/\mathcal{K})} \neq 1,$$

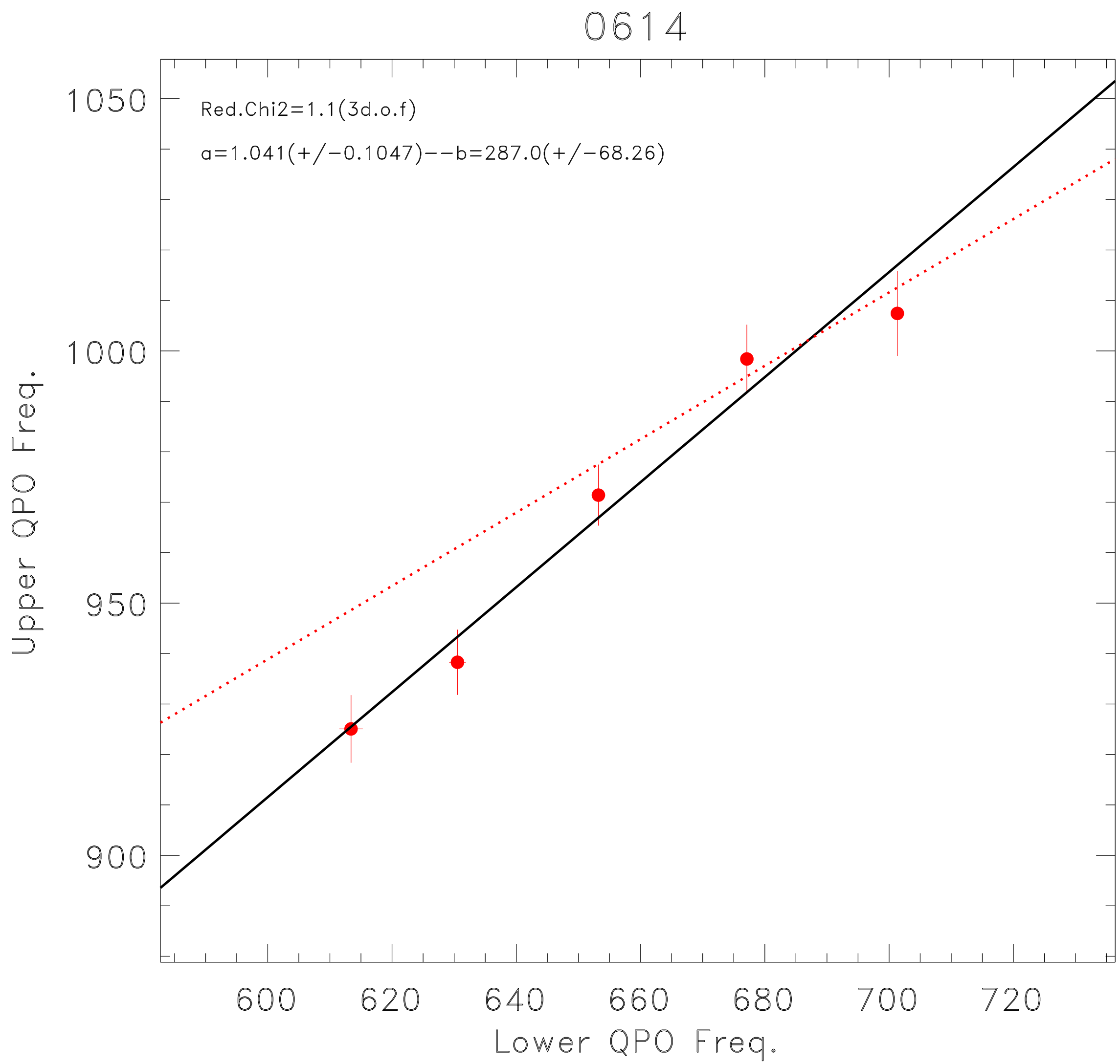
$$B = -A\,\nu^0_{\mathrm{D}}\,(1 + \mathcal{E}\,C_{DD}/\mathcal{K}) + \nu^0_{\mathrm{U}}\,(1 + \mathcal{E}\,C_{UD}/\mathcal{K}).$$

Note: $\Gamma \neq 1 \rightarrow$ slopes of Bursa lines differ from $3/2$ by a *finite* amount.

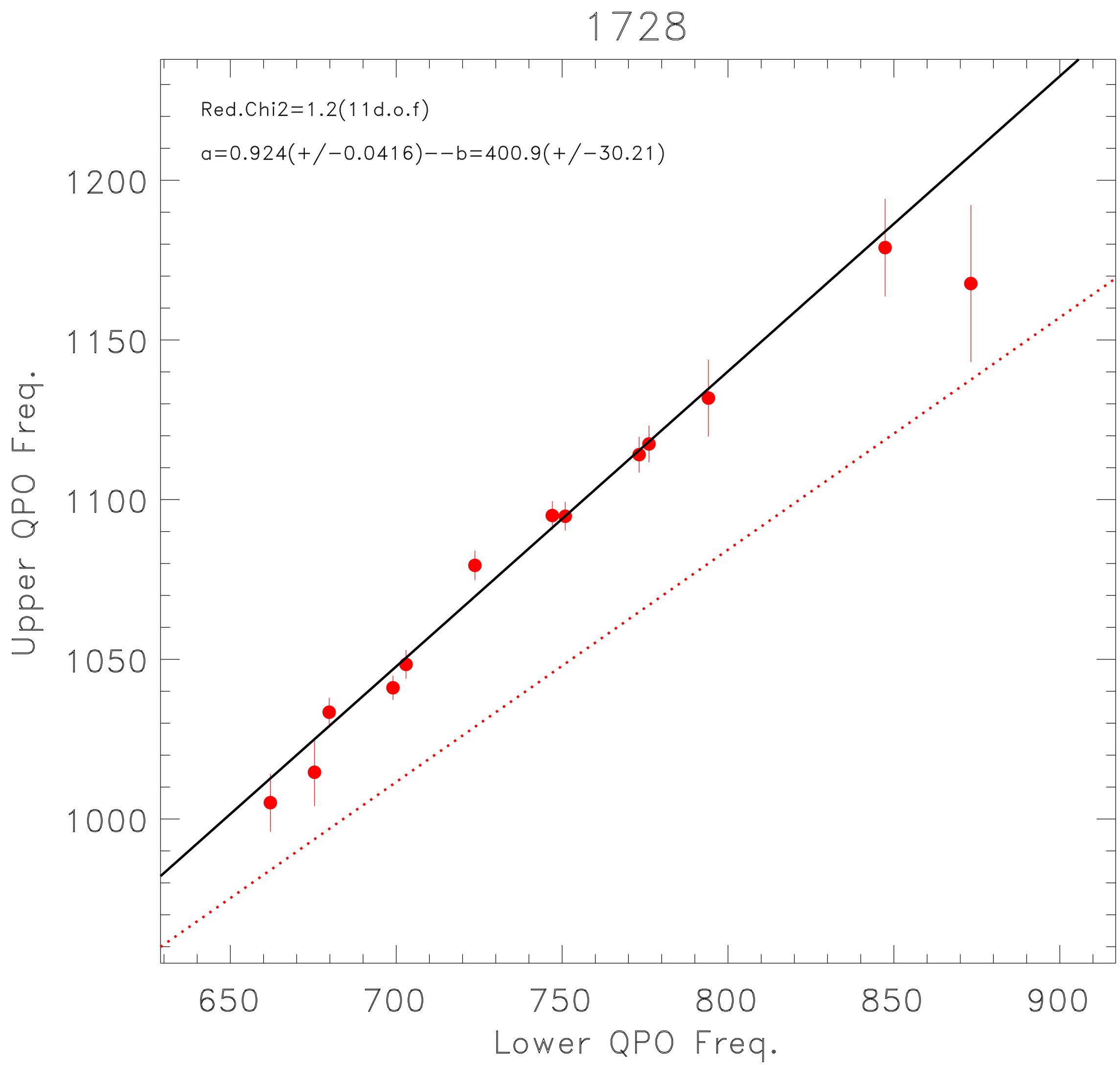
Bursa lines for six neutron stars (0)



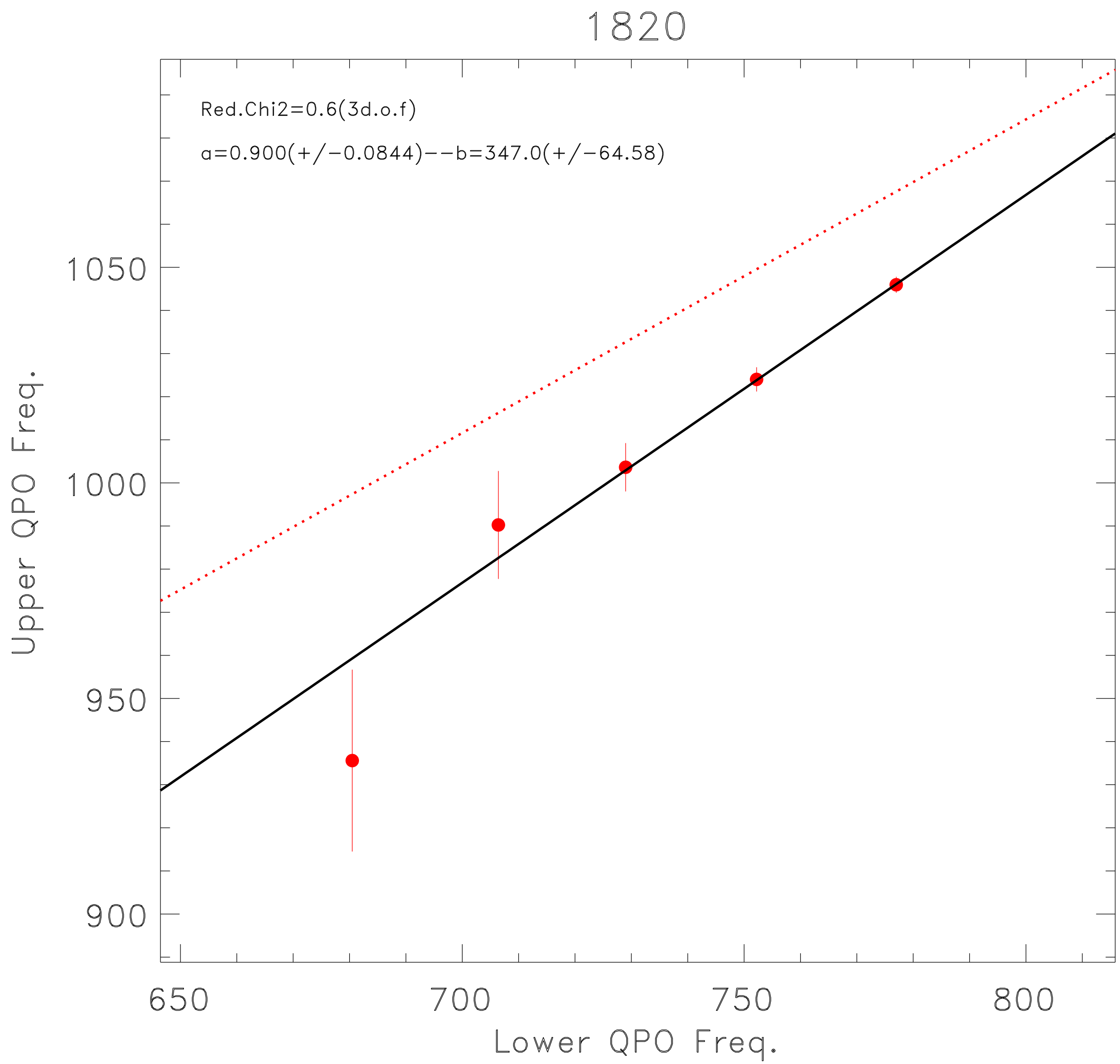
Bursa lines for six neutron stars (1)



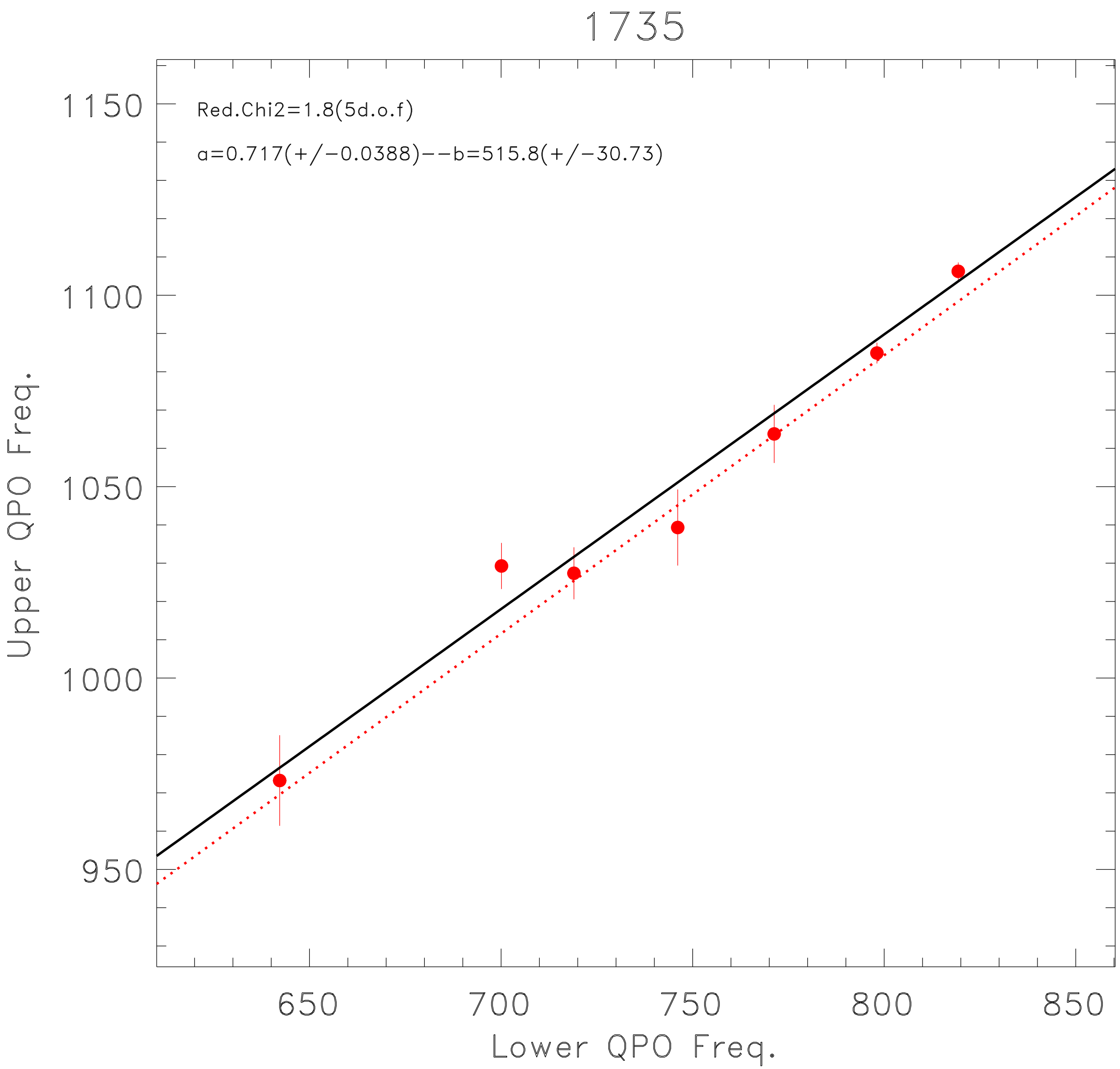
Bursa lines for six neutron stars (2)



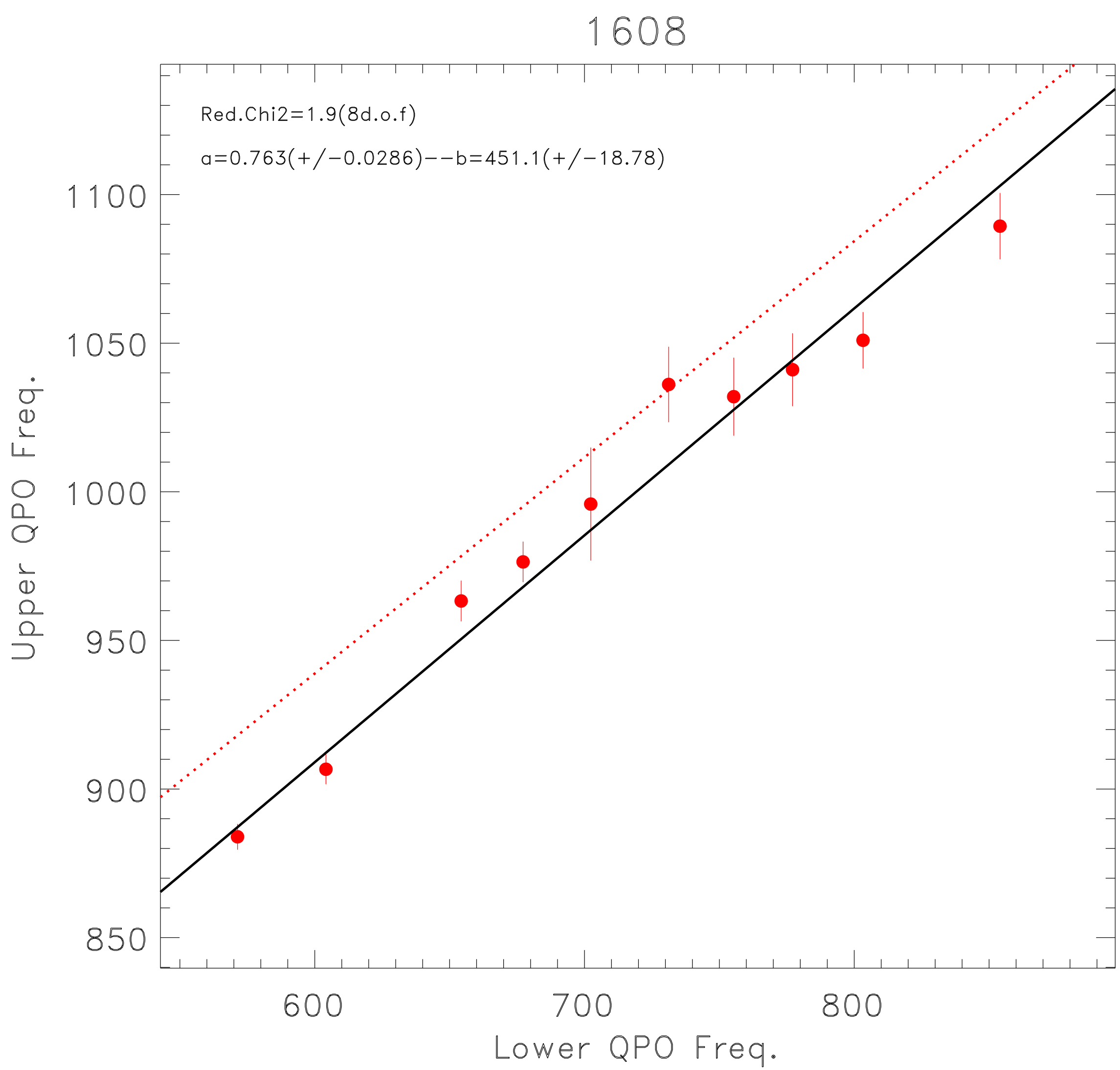
Bursa lines for six neutron stars (3)



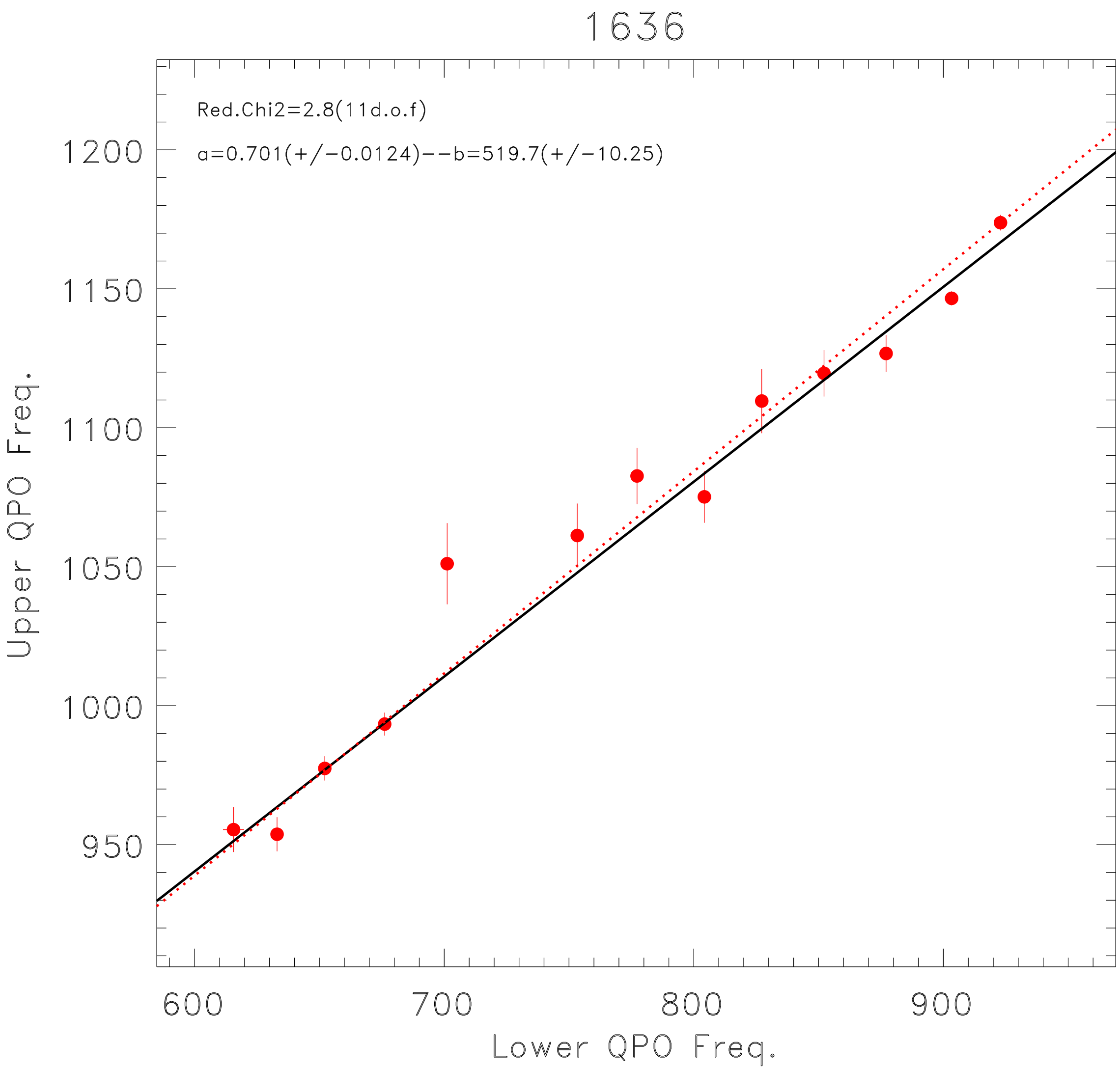
Bursa lines for six neutron stars (4)



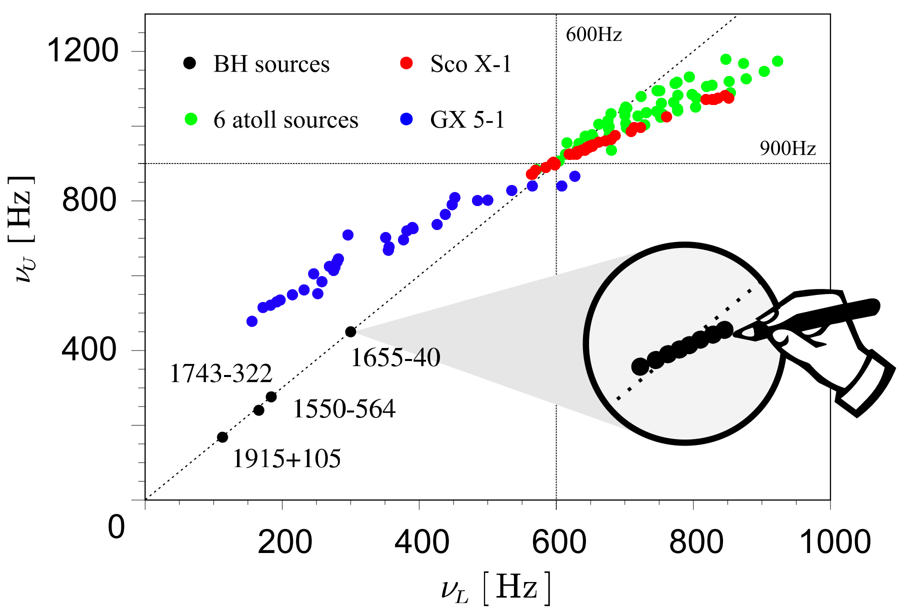
Bursa lines for six neutron stars (5)

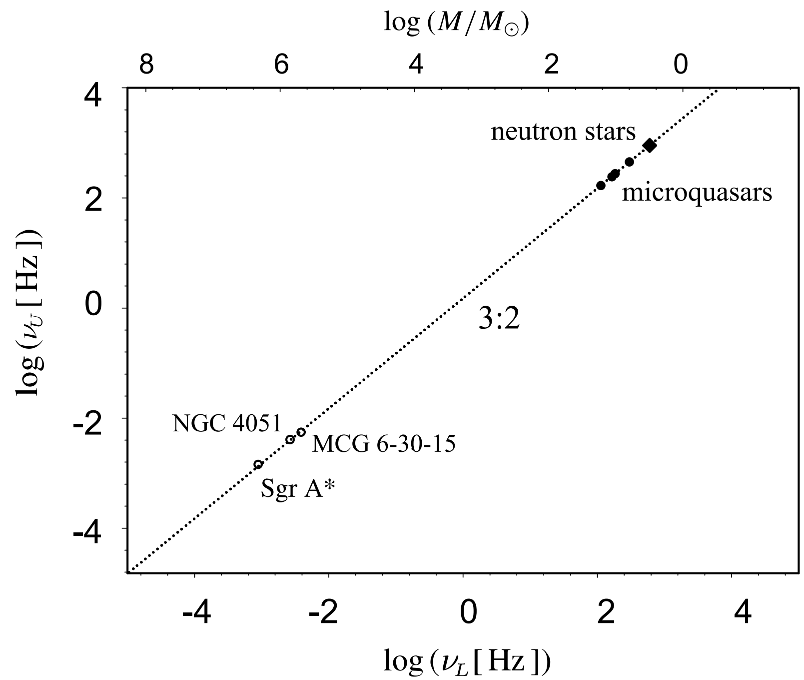


Bursa lines for six neutron stars (6)



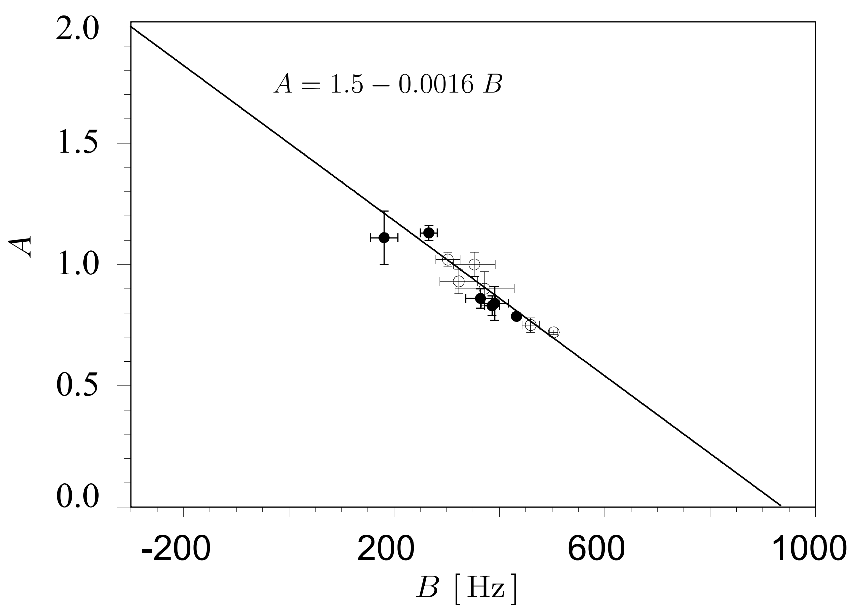
Bursa lines for black holes

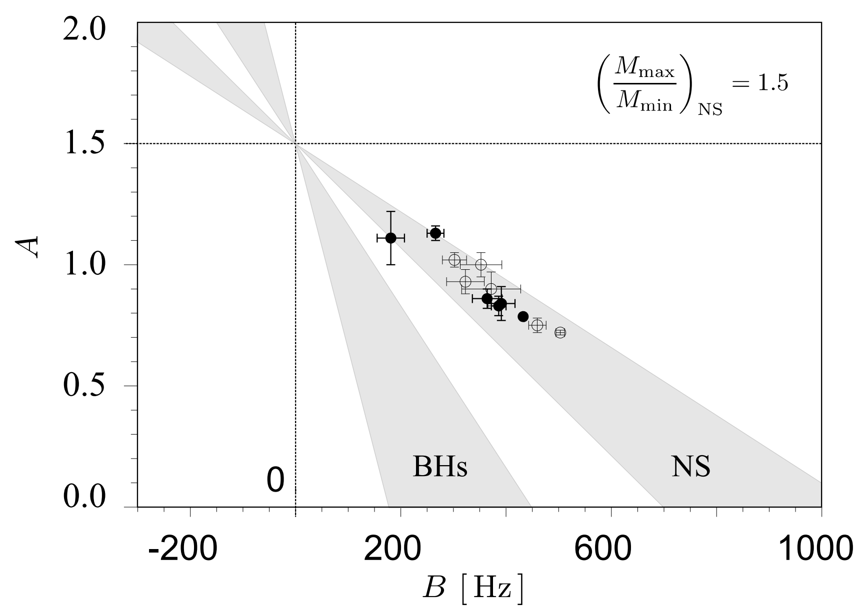




The A-B anti-correlation

$$A = -\gamma\, B \;+\; A_0, \quad \gamma = \frac{1}{\nu^0_{\mathrm{D}}}\left[1 + \mathcal{O}(\alpha^2)\right], \quad A_0 = \frac{\nu^0_{\mathrm{U}}}{\nu^0_{\mathrm{D}}} + \mathcal{O}(\alpha^2).$$





The Atol and Z sources